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Delocalization effects in quasi-1D models with correlated disorder

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Abstract

We introduce a new approach to analyse the global structure of electronic states in quasi-1D models in terms of the dynamics of a system of parametric oscillators with time-dependent stochastic couplings. We thus extend to quasi-1D models the method previously applied to 1D disordered models. Using this approach, we show that a 'delocalization transition' can occur in quasi-1D models with weak disorder with long-range correlations.

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1. Introduction

In recent years there has been a steadily increasing interest in disordered models with long-range correlated disorder. The interest was initially spurred by the discovery that specific long-range correlations can produce a kind of 'delocalization transition' even in one-dimensional (1D) models in which the electronic states are typically exponentially localized [1]. The first numerical results were later confirmed by analytical studies, which identified the relationship existing between localization length and pair correlators of the random potential and showed how to create mobility edges in strictly 1D discrete models [2]. These analytical predictions were experimentally verified by considering the transmission of microwaves in a single-mode waveguide with a random array of correlated scatterers [3]. Later on, the results obtained for discrete lattices [2] were extended to 1D continuous models [4] and applied to related problems such as the propagation of waves in waveguides with random surface scattering and to specific quasi-1D models with bulk scattering [4–6].

An important tool for the analysis of such models is represented by transfer matrix techniques (for a review, see [7]). A related method, which turned out to be very effective, is that based on the analogy between localization phenomena in disordered systems and the dynamics of stochastic oscillators [8, 9]. This dynamical approach was originally applied to strictly 1D models; subsequent research, however, has begun to explore models of higher

dimensionality, with quasi-1D models providing the first natural extension towards realistic disordered systems. A rigorous mathematical treatment of discrete quasi-1D models was given in [10], but the analysis was centred on disorder without spatial correlations. Quasi-1D models with short-range correlated disorder were studied in [11], where the extension of the random dimer model to the quasi-1D case was considered.

This paper serves two main purposes: in the first place, we extend to systems of higher dimensionality the dynamical approach which was used successfully for strictly 1D continuous models (see, e.g., [12] and references therein). We show that the Lyapunov exponents which govern the exponential divergence of initially nearby trajectories in a classical systems of stochastic oscillators are the same exponents which appear in the transfer-matrix approach for quasi-1D disordered models. This allows us to establish a rigorous correspondence between the quantum phenomenon of localization in quasi-1D disordered models and the orbit instability in classical systems of parametric oscillators with noisy couplings.

The second objective of this work is to use this analogy to study the effects of long-range correlations of the disorder on the localization of the electronic states in quasi-1D models. Our main result is that, for weak disorder, specific long-range correlations can make all Lyapunov exponents vanish (within the second-order approximation), thereby suppressing the orbit instability on the one hand and producing a 'delocalization transition' on the other.

The paper is organized as follows. In section 2 we discuss the correspondence of quasi-1D disordered models with a set of coupled parametric oscillators. In section 3 we show how the evolution of this dynamical system can be analysed. The general results thus obtained are then applied in section 4 to the specific case in which the random potential depends only on the longitudinal coordinate. In section 5 we apply the dynamical approach to the case of a generic weak disorder. We determine an expression for the sum of the positive Lyapunov exponents of the quasi-1D model and we use this result to discuss the delocalization effects that are produced by specific long-range correlations of the random potential. The conclusions are then outlined in section 6.

2. Classical representation of the quasi-1D model

2.1. The 1D case

Before considering quasi-1D models, we summarize shortly the main results for the strictly 1D case. Since this work is focused on quasi-1D models, we shall be brief; the interested reader can find more details in [8, 9, 13]. The correspondence between Anderson localization in 1D models with weak disorder and the energetic instability of oscillators with a frequency perturbed by a noise is a straightforward consequence of the mathematical analogy between the Schrödinger equation

$$-\psi''(x) + U(x)\psi(x) = E\psi(x), \tag{1}$$

with the positive energy E and the dynamical equation of a stochastic oscillator

$$\ddot{q}(t) + (\omega^2 - U(t))q(t) = 0. \tag{2}$$

(Here and in the following we will use energy units such that $\hbar^2/2m = 1$). In fact, equation (1) can be easily transformed into equation (2) by interpreting the spatial coordinate x as the time t and the wavefunction amplitude ψ as the coordinate of an oscillator. In the first equation the function U plays the role of a random potential while in the second it represents a noise. The noise is white or coloured depending on whether the disorder is spatially correlated or not.

The mathematical identity of equations (1) and (2) allows one to study the global structure of the quantum eigenstates of the disordered model (1) by analysing the dynamics of the

corresponding classical oscillator (2). The dynamical equation (2) gives the time evolution of the oscillator coordinate $q(t)$ once the initial position $q(0)$ and velocity $\dot{q}(0)$ have been specified; use of analogous boundary conditions to solve the Schrödinger equation (1) leads to the solution $\psi(x)$ which is obtained with the standard transfer matrix approach. Studying the disordered model (1) in terms of the dynamics of the random oscillator (2), therefore, is equivalent to using transfer matrix methods.

Comparing the solutions of equations (1) and (2) one finds that spatially extended states correspond to bounded oscillator orbits, while localized states have their counterpart in unbounded trajectories. As a consequence, the phenomenon of Anderson localization corresponds to the energetic instability of the parametric oscillator, with the inverse localization length being equal to the Lyapunov exponent of the stochastic oscillator, i.e., the rate of exponential divergence for initially nearby orbits.

The case of weak noise/disorder can be studied using perturbative techniques which were originally devised for the study of stochastic systems. These methods allow one to obtain the rate of energy growth for the oscillator (2) and therefore the inverse localization length for the disordered model (1) [9, 13]. The second-order expression for the inverse localization length is

$$\lambda = \frac{1}{4\omega^2} \int_0^\infty \langle U(t)U(t+\tau)\rangle \cos(2\omega\tau)\, \mathrm{d}\tau, \tag{3}$$

which shows that the Lyapunov exponent is proportional to the power spectrum of the disorder, i.e., to the cosine Fourier transform of the two-point correlator of the random potential (here and in what follows we use the symbol $\langle \ldots \rangle$ to denote the average over different disorder realizations). This entails that the energetic instability of the oscillator (2) can be suppressed if the unperturbed oscillator frequency, multiplied by a factor two, lies in a frequency interval where the power spectrum of the disorder vanishes. Correspondingly, delocalized states arise for the energy values for which the inverse localization length (3) vanishes.

2.2. The quasi-1D model

We analyse the phenomenon of electronic localization in a semi-infinite strip. The present method can be applied to bars as well; we focus on strips to avoid unnecessary complications in the mathematical formulae. We consider the strip D in the $x - y$ plane

$$D = \{(x, y) : 0 \leqslant x; 0 \leqslant y \leqslant L\}. \tag{4}$$

The Schrödinger equation for a quantum particle ('electron') in the strip is

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x, y) + \varepsilon U(x, y)\psi(x, y) = E\psi(x, y). \tag{5}$$

The function $U(x, y)$ is the random potential felt by the electrons; the potential can exhibit spatial correlations. The dimensionless parameter ε is introduced to keep track of the order of perturbative expansions and can be set equal to 1 in the final formulae. We will focus on the case of weak disorder, i.e., on the case $\varepsilon \ll 1$. For the model to be completely defined, one must provide the statistical properties of the random potential; for weak disorder, it is enough to specify the first two moments of $U(x, y)$. We will assume that the potential has zero average, $\langle U(x, y)\rangle = 0$, and that the two-point correlator is a known function:

$$\langle U(x_1, y_1)U(x_2, y_2)\rangle = \sigma^2 \chi(x_1 - x_2, y_1 - y_2), \tag{6}$$

where σ^2 represents the variance of the potential

$$\sigma^2 = \langle U(x, y)U(x, y)\rangle$$

provided that it is finite; when the variance diverges, like in the case of white noise, σ^2 must be interpreted as a parameter which measures the strength of the disorder. We will consider random potentials which are translationally invariant in the mean; this is why we assume that the binary correlator (6) depends only on the difference of the coordinates of the two points (x_1, y_1) and (x_2, y_2).

2.3. Representation of the Schrödinger equation in the partially reciprocal space

If one wishes to extend to quasi-1D models the dynamical approach described in subsection 2.1 for the 1D case, it is quite natural to identify the longitudinal coordinate x with the time t of the corresponding dynamical system. One can then Fourier transform the Schrödinger equation (5) in the transversal direction y. In the following we consider the real solutions of the stationary Schrödinger equation (5); this choice is not restrictive because one can find a complete set of real eigenfunctions of the Hamiltonian. It is possible to expand any real function of y defined over the interval $[0 : L]$ in a cosine basis. In this way one avoids dealing with complex Fourier components, which makes easier the successive identification of the Fourier components with oscillator coordinates discussed below. Expanding the wavefunction and the potential one obtains

$$\psi(x, y) = \tilde{\psi}_0(x) + 2\sum_{n=1}^{\infty} \tilde{\psi}_n(x) \cos\left(\frac{\pi n y}{L}\right) = \sum_{n=-\infty}^{\infty} \tilde{\psi}_n(x) \cos\left(\frac{\pi n y}{L}\right) \tag{7}$$

$$U(x, y) = \tilde{U}_0(x) + 2\sum_{n=1}^{\infty} \tilde{U}_n(x) \cos\left(\frac{\pi n y}{L}\right) = \sum_{n=-\infty}^{\infty} \tilde{U}_n(x) \cos\left(\frac{\pi n y}{L}\right), \tag{8}$$

where the Fourier components of the expansions are defined by

$$\tilde{\psi}_n(x) = \frac{1}{L}\int_0^L \mathrm{d}y\, \psi(x, y) \cos\left(\frac{\pi n y}{L}\right) \tag{9}$$

and

$$\tilde{U}_n(x) = \frac{1}{L}\int_0^L \mathrm{d}y\, U(x, y) \cos\left(\frac{\pi n y}{L}\right). \tag{10}$$

In the following we will often refer to the Fourier components of the wavefunctions as Fourier modes or channels. Note that the Fourier components with opposite values of the index n are identical: $\tilde{\psi}_n(x) = \tilde{\psi}_{-n}(x)$ and $\tilde{U}_n(x) = \tilde{U}_{-n}(x)$. The functions defined by the Fourier cosine expansions (7) and (8) coincide with the original functions in the domain (4); outside of this domain, they are symmetric and $2L$-periodic functions of the transversal coordinate y, i.e.,

$$\psi(x, y) = \psi(x, -y) \qquad \text{and} \qquad \psi(x, y) = \psi(x, y + 2L).$$

The same relations also hold for $U(x, y)$. Expanding in a cosine series, therefore, is equivalent to considering y-symmetric functions $\psi(x, y)$ and $U(x, y)$ on the doubled strip $[0 : \infty] \times [-L : L]$ and impose periodic boundary conditions along the transversal direction (i.e., to roll the doubled strip into a cylinder). Our choice of periodic boundary conditions is, in fact, due to some mathematical simplifications. As is typically assumed in the literature, in the case of large number of channels the global properties of transport, determined by the localization length of eigenstates, are insensitive to the type of boundary conditions.

Note that by identifying the wavefunction $\psi(x, y)$ with its Fourier expansion (7) we are neglecting the isolated points where the identity (7) may not hold. Under this assumption, by differentiating twice with respect to y both sides of equation (7), one obtains the expression

$$\frac{\partial^2\psi}{\partial y^2} = -\sum_{n=-\infty}^{\infty}\left(\frac{\pi n}{L}\right)^2 \tilde{\psi}_n(x)\cos\left(\frac{\pi n y}{L}\right).$$

Using this relation and the fact that the Fourier transform of a product can be written in the form of a convolution of the Fourier components, one can write equation (5) in Fourier representation as

$$\frac{\partial^2\tilde{\psi}_n}{\partial x^2}(x) + \left[E - \left(\frac{\pi n}{L}\right)^2\right]\tilde{\psi}_n(x) = \sum_{k=-\infty}^{\infty}\varepsilon\tilde{U}_{n-k}(x)\tilde{\psi}_k(x). \tag{11}$$

2.4. Elliptic and hyperbolic Fourier components

Before giving a dynamical interpretation for equation (11), it is useful to introduce a distinction between 'elliptic' and 'hyperbolic' Fourier components of the wavefunction. Adopting the terminology of [10], we define the Fourier component $\tilde{\psi}_n(y)$ with wave number n to be

$$\text{elliptic} \qquad \text{if} \quad E - \left(\frac{\pi n}{L}\right)^2 > 0$$

$$\text{hyperbolic} \qquad \text{if} \quad E - \left(\frac{\pi n}{L}\right)^2 < 0$$

(we ignore the marginal case of the 'parabolic' components with $E = (\pi n/L)^2$).

The crucial difference between elliptic and hyperbolic components is that the latter decay exponentially in the longitudinal direction: from a physical point of view, they are evanescent modes (or closed channels). Hence they can be neglected for large values of the longitudinal coordinate x. The irrelevance of the hyperbolic modes can be justified also with a different argument. As can be seen from the Fourier expansion (7), the Fourier components with large values of n describe the behaviour of the wavefunction on small spatial scales

$$\delta y \sim \frac{L}{n}$$

in the transversal direction. If we suppose that our continuous model has an underlying lattice structure, however, we need not consider the behaviour of the wavefunction over spatial scales $\delta y \lesssim a$, where a is the lattice constant. This implies that, if the short-scale spatial structure of the wavefunction is neglected, the only Fourier components that must be taken into account are the ones with wavenumber

$$n < \overline{n} \sim \frac{L}{a}. \tag{12}$$

Now, let us restrict our analysis of localization to the case of high-energy electrons, where high-energy means

$$E > \left(\frac{\pi\overline{n}}{L}\right)^2 \sim \left(\frac{\pi}{a}\right)^2. \tag{13}$$

In this case, one can see that, due to the cutoff condition (12), the only Fourier components which are relevant for the analysis are the elliptic ones. Hence, we need not bother with hyperbolic components if we restrict our attention to the regime of sufficiently high energies. This back-of-the-envelope criterion suggests that our approximation may fail when the electron energy does not satisfy condition (13).

After eliminating the hyperbolic components, the Schrödinger equation (11) takes the form

$$\frac{\partial^2 \tilde{\psi}_n}{\partial x^2}(x) + \left[E - \left(\frac{\pi n}{L}\right)^2\right] \tilde{\psi}_n(x) = \sum_{k=-N}^{N} \varepsilon \tilde{U}_{n-k}(x) \tilde{\psi}_k(x), \tag{14}$$

where the indices n and k are restricted to the elliptic modes (or conducting channels), i.e., the indices take the values $-N, -N+1, \ldots, N-1, N$ with N being the integer part $[\ldots]$ of the ratio $L\sqrt{E}/\pi$,

$$N = \left[\frac{L\sqrt{E}}{\pi}\right].$$

2.5. Dynamical equations for the quasi-1D model

In order to give a dynamical interpretation of the Schrödinger equation (14) we replace the longitudinal variable x with a time variable t. In addition, we define the frequencies

$$\omega_n = \sqrt{E - \left(\frac{\pi n}{L}\right)^2} \tag{15}$$

and we introduce the notation

$$q_n(t) = \tilde{\psi}_n(t)$$

for the Fourier components of the wavefunction. This allows us to write equation (14) in the form

$$\ddot{q}_n(t) + \omega_n^2 q_n(t) = \varepsilon \sum_{k=-N}^{N} \tilde{U}_{n-k}(t) q_k(t), \tag{16}$$

which is naturally interpreted as the dynamical equation of a system of classical parametric oscillators with time-dependent stochastic couplings. This correspondence allows one to analyse the structure of the electronic states of the disordered model (5) in terms of the dynamics of the system (16). In fact, the spatial behaviour of the $\tilde{\psi}_n(x)$ Fourier component along the longitudinal direction is determined by the time evolution of the coordinate $q_n(t)$ of the corresponding oscillator.

The system of $2N+1$ second-order differential equations (16) can be transformed into a system of $4N+2$ first-order Hamiltonian equations by introducing the momenta $p_n = \dot{q}_n$ and the Hamiltonian

$$H(p,q) = \sum_{n=-N}^{N} \left(\frac{p_n^2}{2} + \frac{\omega_n^2}{2} q_n^2\right) - \varepsilon \sum_{n=-N}^{N} \sum_{k=-N}^{N} \tilde{U}_{n-k}(t) q_n q_k. \tag{17}$$

Then one can cast the dynamical system (16) in the form

$$\begin{aligned} \dot{p}_n &= -\frac{\partial H}{\partial q_n} = -\omega_n^2 q_n + \varepsilon \sum_{k=-N}^{N} \tilde{U}_{n-k}(t) q_k \\ \dot{q}_n &= \frac{\partial H}{\partial p_n} = p_n. \end{aligned} \tag{18}$$

Note that these Hamiltonian equations describe a system of $2N+1$ oscillators. However, not all oscillators are independent, because the symmetry of the Fourier components $\tilde{\psi}_n(x) = \tilde{\psi}_{-n}(x)$ implies that the Hamiltonian system is subject to the constraints $q_n = q_{-n}$

which reduce to $N+1$ the number of degrees of freedom of the system. Making use of the constraints, the dynamical equations (18) can be written in the form

$$\dot{p}_n = -\omega_n^2 q_n + \sum_{k=0}^{N} \varepsilon W_{nk}(t) q_k \qquad \dot{q}_n = p_n, \tag{19}$$

where $n = 0, 1, \ldots, N$ and only independent oscillators are involved. Note that in equation (19) we have introduced the short-hand notation

$$W_{n,k}(t) = [\tilde{U}_{n+k}(t) + \tilde{U}_{n-k}(t)]\left(1 - \tfrac{1}{2}\delta_{k0}\right). \tag{20}$$

A comment is in order here. A mapping of a quasi-1D disordered model of the form (5) unto a Hamiltonian system was made long ago by Hansel and Luciani [14, 15]. In spite of several similarities, their approach differs from the present one in many aspects. In the first place, in [14, 15], the quasi-1D model is morphed into a Hamiltonian system by *discretizing* rather than *Fourier transforming* the system in the transversal directions. In our approach, no discretization is required. This implies that the channels Hansel and Luciani deal with are 1D chains coupled linearly to each other, whereas in our scheme the channels are the Fourier modes of the wavefunction. Because of this conceptual difference, our channels are coupled only if the random potential is present, in contrast to what happens in the Hansel and Luciani work. Similarly, no distinction between elliptic and hyperbolic channels arises in that case. In the second place, the Hamiltonian model of Hansel and Luciani has no constraints, while the system (17) is subject to the constraints $q_n = q_{-n}$. This entails that the coupling term (20) is not fully symmetric in the two indices n and k, unlike the coupling matrix considered in [14, 15].

2.6. *Wavefunction localization versus oscillator instability*

The mapping of equation (14) unto equation (19) makes possible to study the behaviour of the wavefunctions of the disordered model (5) in terms of the time evolution of the dynamical system (19). From a physical point of view, the mathematical identity of the two problems translates into a correspondence between the phenomenon of Anderson localization in quasi-1D disordered bars and the energetic instability of the system of random oscillators. The connection between the two phenomena can be understood and made quantitative by observing that the crucial properties of both classes of systems are defined in terms of their Lyapunov exponents. In the case of quasi-1D systems, Lyapunov exponents arise within the framework of the transfer matrix approach (see, e.g., [16] and references therein). The method divides the strip (or the bar) in layers; a transfer matrix is an operator that relates the values of the wavefunction and its derivative on one layer with the corresponding values on the contiguous layer. The Schrödinger equation is considered as an initial value problem and its solution is obtained in terms of a product of transfer matrices (in discrete models) or of spatial-ordered exponentials (for continuous models). Making use of Oseledec's theorem [17], one can then define the Lyapunov exponents for the quasi-1D model in terms of the eigenvalues of the asymptotic product of transfer matrices. As is well known, the localization length in this approach is determined by the smallest Lyapunov exponent (see [18] and references therein).

From this point of view, quasi-1D models are similar to 1D systems of the type described by equation (1), whose single Lyapunov exponent (i.e., the inverse of the localization length) is identical to the Lyapunov exponent of the dynamical counterpart (2) (i.e., the rate of exponential orbit divergence). In quasi-1D models, however, the picture is more complex than in the strictly 1D case because there is not a single Lyapunov exponent, but a whole Lyapunov spectrum.

In dynamical systems, the most significative Lyapunov exponent is the *largest* one, because it defines the rate of exponential divergence of initially nearby orbits along almost every direction of the phase space of the system. Smaller Lyapunov exponents play a reduced dynamical role, because they give the rate of exponential divergence for zero-measure sets of initial conditions [19]. In quasi-1D models, on the other hand, the most relevant Lyapunov exponent is the *smallest* one [18]. This can be seen by considering that the Lyapunov exponents obtained in the transfer matrix approach can be interpreted as rates of exponential decay of the wavefunction; since the spatial extension of the electronic state is determined by the smallest decay rate, the smallest Lyapunov exponent is that which matters most.

In conclusion, quasi-1D models and their dynamical counterparts share the same Lyapunov spectra, but the physical relevance of the single Lyapunov exponents is different for the two classes of systems. Therefore, when the study of the dynamical system (19) is used as a mean to extract information on the spatial behaviour of the electronic states of model (5), one needs to push the dynamical analysis of the system (19) beyond the determination of the largest Lyapunov exponent and to compute the whole Lyapunov spectrum. Having thus highlighted the links between the quantum phenomenon of localization and the dynamics of a system of parametric oscillators, we now turn our attention to the study of the latter problem.

3. Dynamics of the parametric oscillators

To analyse the dynamics of a system of oscillators, it is convenient to perform a canonical transformation and switch from the Cartesian coordinates (q_n, p_n) to the action-angle variables (J_n, θ_n) defined by the relations

$$q_n = \sqrt{\frac{2J_n}{\omega_n}}\sin\theta_n \qquad p_n = \sqrt{2\omega_n J_n}\cos\theta_n.$$

In terms of the new variables the Hamiltonian (17) takes the form

$$H = \sum_{n=-N}^{N}\omega_n J_n - \sum_{n=-N}^{N}\sum_{k=-N}^{N}\varepsilon\tilde{U}_{n-k}(t)\sqrt{\frac{J_n J_k}{\omega_n\omega_k}}\sin\theta_n\sin\theta_k,$$

and the dynamical equations (19) become

$$\begin{aligned}\dot{J}_n &= \sum_{k=0}^{N} 2\varepsilon W_{n,k}(t)\sqrt{\frac{J_n J_k}{\omega_n\omega_k}}\sin\theta_k\cos\theta_n \\ \dot{\theta}_n &= \omega_n - \sum_{k=0}^{N}\varepsilon W_{n,k}(t)\sqrt{\frac{J_k}{J_n\omega_n\omega_k}}\sin\theta_k\sin\theta_n.\end{aligned} \tag{21}$$

Following the general method described by Van Kampen [20], one can replace the system of Langevin equations (21) with a deterministic Fokker–Planck equation for the probability distribution of the stochastic variables (J_n, θ_n). In fact, the dynamical equations (21) are stochastic equations of the form

$$\dot{u} = F^{(0)}(u) + \varepsilon F^{(1)}(u, t), \tag{22}$$

where $F^{(0)}(u)$ is a deterministic function of the vector u and $\varepsilon F^{(1)}(u, t)$ is a stochastic correction to the deterministic term. Following Van Kampen [20], one can associate to the stochastic differential equation (22) an ordinary differential equation for the function $P(u, t)$

which represents the probability distribution for the random variable u. The associated differential equation has the form

$$\frac{\partial P}{\partial t}(u,t) = -\sum_i \frac{\partial}{\partial u_i}\left[F_i^{(0)}(u)P(u,t)\right] + \varepsilon^2 \sum_{ij} \frac{\partial}{\partial u_i}\int_0^\infty \mathrm{d}\tau \left\langle F_i^{(1)}(u,t)\frac{\mathrm{d}(u^{-\tau})}{\mathrm{d}(u)}\frac{\partial}{\partial u_j^{-\tau}}F_j^{(1)}(u^{-\tau},t-\tau)\right\rangle \times \frac{\mathrm{d}(u)}{\mathrm{d}(u^{-\tau})}P(u,t) + o(\varepsilon^2), \tag{23}$$

where u^t is the solution at time t of the unperturbed differential equation $\dot{u} = F^{(0)}(u)$ with the initial condition $u(0) = u$ and $\mathrm{d}(u^{-\tau})/\mathrm{d}(u)$ is the Jacobian of the transformation $u \to u^{-\tau}$. Note that, within the second-order approximation, i.e., neglecting terms of order $o(\varepsilon^2)$, only first- and second-order derivatives appear on the right-hand side (rhs) of equation (23), which therefore has the form of a Fokker–Planck equation.

In the present case, the vectors u, $F^{(0)}(u)$ and $F^{(1)}(u,t)$ have the nth bi-component equal to

$$u_n = \begin{pmatrix} J_n \\ \theta_n \end{pmatrix}, \qquad F_n^{(0)} = \begin{pmatrix} 0 \\ \omega_{\mathbf{n}} \end{pmatrix}, \qquad F_n^{(1)} = \begin{pmatrix} F_{n,J}^{(1)}(J,\theta) \\ F_{n,\theta}^{(1)}(J,\theta) \end{pmatrix},$$

where we have introduced the symbols

$$F_{n,J}^{(1)}(J,\theta) = \sum_{k=0}^{N} 2W_{n,k}(t)\sqrt{\frac{J_n J_k}{\omega_n \omega_k}}\sin\theta_k \cos\theta_n$$

$$F_{n,\theta}^{(1)}(J,\theta) = -\sum_{k=0}^{N} W_{n,k}(t)\sqrt{\frac{J_k}{J_n \omega_n \omega_k}}\sin\theta_k \sin\theta_n.$$

The unperturbed flow $u \to u^t$ in this specific case has the simple form

$$u_n^t = \begin{pmatrix} J_n^t \\ \theta_n^t \end{pmatrix} = \begin{pmatrix} J_n \\ \omega_n t + \theta_n \end{pmatrix}$$

and as a consequence one has

$$\frac{\mathrm{d}(u^{-\tau})}{\mathrm{d}(u)} = \frac{\mathrm{d}(u)}{\mathrm{d}(u^{-\tau})} = 1 \qquad \text{and} \qquad \frac{\partial}{\partial u_i^{-\tau}} = \frac{\partial}{\partial u_i}.$$

Using these results, the general Fokker–Planck equation (23) takes the specific form

$$\frac{\partial P}{\partial t}(J,\theta,t) = -\sum_{n=0}^{N} \omega_n \frac{\partial P}{\partial \theta_n}(J,\theta,t) + \varepsilon^2 \sum_{n=0}^{N}\sum_{k=0}^{N} \frac{\partial}{\partial J_n}\int_0^\infty \mathrm{d}\tau \left\langle F_{n,J}^{(1)}(J,\theta,t)\frac{\partial}{\partial J_k}F_{k,J}^{(1)}(J,\theta-\omega\tau,t-\tau)\right\rangle P(J,\theta,t) + \varepsilon^2 \sum_{n=0}^{N}\sum_{k=0}^{N} \frac{\partial}{\partial J_n}\int_0^\infty \mathrm{d}\tau \left\langle F_{n,J}^{(1)}(J,\theta,t)\frac{\partial}{\partial \theta_k}F_{k,\theta}^{(1)}(J,\theta-\omega\tau,t-\tau)\right\rangle P(J,\theta,t)$$

$$+\varepsilon^2\sum_{n=0}^{N}\sum_{k=0}^{N}\frac{\partial}{\partial\theta_n}\int_0^\infty \mathrm{d}\tau\left\langle F^{(1)}_{n,\theta}(J,\theta,t)\frac{\partial}{\partial J_k}F^{(1)}_{k,J}(J,\theta-\omega\tau,t-\tau)\right\rangle P(J,\theta,t)$$
$$+\varepsilon^2\sum_{n=0}^{N}\sum_{k=0}^{N}\frac{\partial}{\partial\theta_n}\int_0^\infty \mathrm{d}\tau\left\langle F^{(1)}_{n,\theta}(J,\theta,t)\frac{\partial}{\partial\theta_k}F^{(1)}_{k,\theta}(J,\theta-\omega\tau,t-\tau)\right\rangle P(J,\theta,t). \tag{24}$$

Rather than trying to obtain the general solution of equation (24)—an exceedingly difficult task—we aim at deriving the reduced probability distribution for the action variables

$$P(J,t)=\int\prod_n \mathrm{d}\theta_n\, P(J,\theta,t).$$

Since the frequencies ω_n are different from zero, a glance at the dynamical equations (21) shows that the angular variables are 'fast' in comparison with the 'slow' action variables. We can therefore assume that, after a sufficiently long time, the angle variables become uncorrelated random variables with a uniform distribution in the $[0:2\pi]$ interval, i.e., that for large times the distribution $P(J,\theta,t)$ reduces to the factorized form

$$P(J,\theta,t)\simeq\frac{1}{(2\pi)^{N+1}}P(J,t). \tag{25}$$

Substituting the distribution (25) in the Fokker–Planck equation (24) and integrating over the angular variables, one obtains, after some algebra, the reduced Fokker–Planck equation

$$\frac{\partial P}{\partial t}(J,t)=\sum_{n=0}^{N}\sum_{k=0}^{N}\frac{1}{2}\frac{\partial}{\partial J_n}\left[D_{nk}(J)\frac{\partial P}{\partial J_k}(J,t)\right], \tag{26}$$

where the diffusion matrix $D_{nk}(J)$ has diagonal elements defined as

$$D_{nn}(J)=\left(\frac{\varepsilon J_n}{\omega_n}\right)^2\int_0^\infty \mathrm{d}\tau\,\langle W_{nn}(t)W_{nn}(t-\tau)\rangle\cos 2\omega_n\tau$$
$$+\sum_{k\neq n}\frac{\varepsilon^2 J_n J_k}{\omega_n\omega_k}\int_0^\infty \mathrm{d}\tau\,\langle W_{nk}(t)W_{nk}(t-\tau)\rangle[\cos(\omega_n+\omega_k)\tau+\cos(\omega_n-\omega_k)\tau] \tag{27}$$

and off-diagonal elements ($n\neq k$) equal to

$$D_{nk}(J)=\frac{\varepsilon^2 J_n J_k}{\omega_n\omega_k}\int_0^\infty \mathrm{d}\tau\,\langle W_{nk}(t)W_{kn}(t-\tau)\rangle[\cos(\omega_n+\omega_k)\tau-\cos(\omega_n-\omega_k)\tau]. \tag{28}$$

The Fokker–Planck equation (26) is a rather general result, because it has been obtained for a generic random potential, with the only assumption that the disorder should be weak and translationally invariant in the mean. We note that only second-order derivatives appear in the diffusion equation (26), and that the coefficients of its rhs are completely determined by the two-point correlator of the random potential. Both facts are consequences of the decision to study the problem within the framework of the second-order approximation. Pushing the perturbative approach to higher orders, in fact, would make higher order derivatives appear in the diffusion equation, together with coefficients which would depend on higher order moments of the potential. It may be appropriate to stress that, if in the present work the statistical properties of disorder are defined only up to the binary correlator, this is not due to any Gaussian assumption but is only a consequence of the adopted second-order perturbative scheme, which makes unnecessary to specify the statistical features of the random potential beyond the second moments.

Using the Fokker–Planck equation (26) for the reduced probability density $P(J, t)$ allows one to obtain the dynamical equation for the averaged action variables

$$\overline{J_n} = \int dJ_0 \cdots dJ_N J_n P(J_0, \ldots, J_N, t).$$

In fact, differentiating with respect to time both members of the previous equation and using equation (26) to express the time derivative of $P(J, t)$, one arrives at the matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{J_0}\\ \vdots \\ \overline{J_N}\end{pmatrix} = \mathbf{M}\begin{pmatrix}\overline{J_0}\\ \vdots \\ \overline{J_N}\end{pmatrix}, \tag{29}$$

where $\mathbf{M}$ is a $(N+1)\times(N+1)$ matrix with diagonal elements

$$\mathbf{M}_{nn} = \frac{\varepsilon^2}{\omega_n^2}\int_0^\infty \mathrm{d}\tau \langle W_{nn}(t) W_{nn}(t-\tau)\rangle \cos 2\omega_n\tau + \sum_{k\neq n}\frac{\varepsilon^2}{2\omega_n\omega_k}\int_0^\infty \mathrm{d}\tau \langle W_{nk}(t) W_{kn}(t-\tau)\rangle[\cos(\omega_n+\omega_k)\tau - \cos(\omega_n-\omega_k)\tau]$$

and off-diagonal elements $(n \neq k)$

$$\mathbf{M}_{nk} = \frac{\varepsilon^2}{2\omega_n\omega_k}\int_0^\infty \mathrm{d}\tau \langle W_{nk}(t) W_{nk}(t-\tau)\rangle[\cos(\omega_n+\omega_k)\tau + \cos(\omega_n-\omega_k)\tau].$$

An important consequence of equation (29) is that, except in very special cases (discussed in section 4), the exponential rate of energy growth is the same for all oscillators. In fact, every action variable increases exponentially in time with a rate given by the largest eigenvalue of the $\mathbf{M}$ matrix.

4. Longitudinal disorder

As a first application of the previous results, we can consider the special case in which the random potential depends only on the longitudinal coordinate, i.e., $U(x, y) = U(x)$. We will refer to this case as 'longitudinal disorder'. This form of random potential has been considered in a different context, i.e., that of many-mode waveguides with a rough surface, where it has been christened as 'stratified disorder' [21]. Here we recover the results of that paper using our more general formalism.

In the special case in which U depends only on the longitudinal coordinate x, the matrix elements (20) take the simple form

$$W_{nk} = U(t)\delta_{nk}. \tag{30}$$

Substituting this expression in equations (27) and (28), one obtains that the diffusion matrix becomes

$$D_{nk}(J) = 4\lambda_n J_n^2 \delta_{nk},$$

where we have introduced the symbols

$$\lambda_n = \frac{\varepsilon^2}{4\omega_n^2}\int_0^\infty \mathrm{d}\tau \langle U(t)U(t+\tau)\rangle \cos 2\omega_n\tau. \tag{31}$$

As a consequence, the Fokker–Planck equation (26) reduces to

$$\frac{\partial P}{\partial t}(J, t) = \sum_{n=0}^{N} 2\lambda_n \frac{\partial}{\partial J_n}\left[J_n^2 \frac{\partial P}{\partial J_n}(J, t)\right].$$

The solution of this Fokker–Planck equation corresponding to the initial distribution

$$P(J, t = 0) = \prod_{n=0}^{N} \delta(J_n - J_n(0))$$

is a product of log-normal distributions

$$P(J, t) = \prod_{n=0}^{N} \frac{1}{\sqrt{8\pi\lambda_n t}} \exp\left\{-\frac{[\log(J_n/J_n(0)) - 4\lambda_n t]^2}{8\lambda_n t}\right\}. \tag{32}$$

Using expression (30) one can also compute the matrix elements of the **M** matrix which determines, via equation (29), the time evolution of the averaged action variables. In the case of longitudinal disorder the **M** matrix takes the simple diagonal form

$$\mathbf{M}_{nk} = 4\lambda_n \delta_{nk}.$$

Both this result and the factorized distribution (32) imply that, in the case of longitudinal disorder, the parametric oscillators are decoupled. The result could have been obtained *a priori* by noting that if the random potential depends only on the longitudinal coordinate, Fourier transforming the Schrödinger equation (5) gives a system of independent equations for the Fourier components. Thus, the quasi-1D model (5) is effectively decomposed into $N + 1$ strictly 1D systems, in agreement with the results of [21].

Besides being independent, the oscillators are also energetically unstable; in fact, solving equation (29) one obtains

$$\overline{J_n} = \mathrm{e}^{4\lambda_n t} J_n(0).$$

The coefficients $4\lambda_n$ are therefore the mean rates of the exponential growth of the energies of the oscillators (since the energy of the nth oscillator is proportional to the corresponding action variable, $E_n = \omega_n J_n$). Taking into account that the rate of exponential increase of the energy is four times the rate of exponential orbit divergence [9], we are led to the conclusion that the coefficients (31) represent the Lyapunov spectrum of the dynamical system (19) and, consequently, of the quasi-1D model (5). The localization length, therefore, is the equal to the inverse of the minimum Lyapunov exponent. Which of the Lyapunov exponents (31) is the smallest depends on the specific form of the Fourier transform of the binary correlator $\langle U(t)U(t+\tau)\rangle = \sigma^2\chi(\tau)$.

In the case of δ-correlated disorder, the Fourier transform of $\chi(\tau) = \delta(\tau)$ is simply the unity, $\tilde{\chi}(\omega) = 1$, and the smallest Lyapunov exponent corresponds to the oscillator with largest unperturbed frequency, i.e., the oscillator with $n = 0$. Hence the inverse of the localization length is

$$l^{-1} = \lambda_0 = \frac{\varepsilon^2\sigma^2}{8\omega_0^2} = \frac{\varepsilon^2\sigma^2}{8E}.$$

When the binary correlation function is not a delta, however, the smallest Lyapunov exponent is not determined only by the largest frequency, but also by the behaviour of the power spectrum of the random potential:

$$\tilde{\chi}(\omega) = \int_{-\infty}^{\infty} \chi(\tau)\cos\omega\tau\,\mathrm{d}\tau = \frac{1}{\sigma^2}\int_{-\infty}^{\infty} \langle U(t+\tau)U(t)\rangle \cos\omega\tau\,\mathrm{d}\tau. \tag{33}$$

An important consequence is that the system can go through a delocalization transition if long-range correlations of the disorder make the Fourier transform of two-point correlator (33) vanish in a specific frequency interval. This phenomenon has already been analysed for 1D systems [2, 9, 13] and for quasi-1D waveguides with stratified disorder [21].

The main point is that, given a Lyapunov exponent $\lambda_n(\omega)$ with *any* specific dependence on the frequency ω, it is possible to find a random potential $U(t)$ that generates the pre-defined function $\lambda_n(\omega)$. Specifically, if the Lyapunov exponent $\lambda_n(\omega)$ is known, then the power spectrum $\tilde{\chi}(\omega) = 8\omega^2\lambda_n(\omega)/\sigma^2$ is also defined. One can then determine a function $\beta(t)$ whose Fourier transform is $\sqrt{\tilde{\chi}(\omega)}$ via the inversion formula

$$\beta(t) = \int_{-\infty}^{\infty} \sqrt{\tilde{\chi}(\omega)}\, \mathrm{e}^{-\mathrm{i}\omega t} \frac{\mathrm{d}\omega}{2\pi}.$$

A random potential which produces the desired behaviour of $\lambda_n(\omega)$ is then obtained by taking the convolution of the function $\beta(t)$ with a stochastic process $\eta(t)$ with zero mean and delta-shaped correlation function. In other words, one can consider the potential

$$U(t) = \int_{-\infty}^{+\infty} \mathrm{d}s\, \beta(s)\eta(t+s),$$

where $\eta(t)$ is a white noise with

$$\langle \eta(t) \rangle = 0 \qquad \text{and} \qquad \langle \eta(t)\eta(t+\tau) \rangle = \delta(\tau).$$

Following this recipe, one can obtain for instance a random potential with the long-ranged correlation function

$$\chi(\tau) = \frac{1}{\tau}(\sin \nu_1 \tau - \sin \nu_2 \tau), \tag{34}$$

which corresponds to the 'window' power spectrum

$$\tilde{\chi}(\omega) = \begin{cases} 1 & \text{if } \nu_1 < \omega < \nu_2 \\ 0 & \text{otherwise.} \end{cases}$$

In such a case, one has that for the every frequency ω_n that falls outside of the interval $[\nu_1/2 : \nu_2/2]$ the corresponding Lyapunov exponent λ_n vanishes (at least within the limits of the second-order approximation considered here). By shifting the frequencies ν_1 and ν_2, one can therefore obtain a delocalization transition as soon as the smallest Lyapunov exponent vanishes; this corresponds to the electronic wavefunction having one extended Fourier component. If more Lyapunov exponents vanish, the number of extended Fourier components increases and the delocalization effect becomes more robust; in the extreme case when all Lyapunov exponents vanish, the electronic wavefunction is not affected by the random potential. If the phenomenon is considered from the point of view of the dynamical system (19) one has that for every vanishing Lyapunov exponent there is an oscillator which becomes stable energywise. When all the Laypunov exponents are zero, all the oscillators are stabilized, i.e., the exponential divergence of the orbits is suppressed and the dynamics ceases to be chaotic. The possibility of making selected Lyapunov exponents vanish entails that in solid-state models the transmission properties can exhibit anomalous and unexpected features of selective transparency, as happens for waveguides. We refer the reader to [21] for a detailed discussion of this phenomenon.

5. Delocalization transition in quasi-1D models

In the previous section we have considered the special case in which the random potential depends only on the longitudinal coordinate. The methods of section 3 can be applied also to a potential of the general form $U = U(x, y)$, but in this case it is usually impossible to find the analytic solution $P(J, t)$ of the Fokker–Planck equation (26) as well as to solve with non-numerical methods the differential equation (29) for the average action variables. In fact,

the dependence of the random potential on the transversal variable produces coupling of the different oscillators, so that the coupling matrix (20) is no longer diagonal. As a consequence, neither the diffusion matrix D_{nk} in the Fokker–Planck equation (26) nor the evolution matrix M_{nk} in equation (29) are diagonal, and this makes exceedingly difficult to solve analytically both equations.

One can, however, use a different technique to determine the *sum* of the positive Lyapunov exponents, or Kolmogorov entropy. Setting the latter equal to zero then gives a sufficient condition for delocalization of the electronic states of the quasi-1D model (5) and a sufficient and necessary condition for the suppression of the orbit instability in the dynamical system (19). In the following subsection we define the Kolmogorov entropy and we show how it can be computed in the second-order approximation. The reader uninterested in technical details may skip this subsection and the next, where a few special cases are analysed, and go to subsection 5.3 where delocalization effects are discussed.

5.1. The Kolmogorov entropy

It is well known that in a Hamiltonian system with ν degrees of freedom there are 2ν Lyapunov exponents which, due to the symplectic structure of the dynamical equations, obey the symmetry relation

$$\lambda_i = -\lambda_{2\nu-i+1}$$

with $i = 1, \ldots, \nu$ (see, for instance, [22]). Because of this relation, one has ν non-negative Lyapunov exponents. For a deterministic Hamiltonian system, at least one of these exponents vanishes; in the present case, however, the presence of a noisy term in the Hamiltonian (17) ensures that, under normal circumstances, the non-negative exponents are actually positive. The Lyapunov exponents are defined as the exponential rate of local divergence of initially nearby trajectories; this definition, however, can be used operatively only to compute the largest Lyapunov exponent. Lesser Lyapunov exponent can be determined using a technique devised by Benettin *et al* [23]. The main idea is that the sum of the largest k exponents is equal to the exponential rate of increase in time of the volume of a parallelepiped spanned by k independent vectors (with $k \leqslant \nu$). The result does not depend on the choice of the initial vectors.

In the present case, we are interested in the sum of all the positive Lyapunov exponents; for this reason we consider the volume of the parallelepiped spanned by $\nu = N + 1$ linearly independent vectors $\vec{\xi}^{(0)}(t), \ldots, \vec{\xi}^{(N)}(t)$. Such a volume can be expressed as the square root of a Gram determinant:

$$V^{(\nu)}(t) = \sqrt{|\det \mathbf{G}(t)|}, \tag{35}$$

where $\mathbf{G}$ is the $(N + 1) \times (N + 1)$ matrix with elements

$$\mathbf{G}_{ij}(t) = \vec{\xi}^{(i)}(t) \cdot \vec{\xi}^{(j)}(t). \tag{36}$$

The sum of the $N + 1$ positive Lyapunov exponents can then be written as

$$\sum_{i=0}^{N} \lambda_i = \lim_{T\to\infty} \left\langle \log \frac{V^{(\nu)}(T)}{V^{(\nu)}(0)} \right\rangle. \tag{37}$$

To apply this prescription to the dynamical system (19) it is convenient to introduce the rescaled variables

$$x_n = \frac{p_n}{\sqrt{\omega_n}} \qquad x_{N+1+n} = \sqrt{\omega_n} q_n,$$

with $n = 0, \ldots, N$, so that the unperturbed motion of the nth oscillator reduces to a rotation in the (x_n, x_{N+1+n}) plane.

In terms of the new variables, the dynamical equations (19) take the form

$$\dot{x}_n = -\omega_n x_{N+1+n} + \sum_{k=0}^{N} \frac{\varepsilon W_{nk}(t)}{\sqrt{\omega_n \omega_k}} x_{N+1+k} \qquad \dot{x}_{N+1+n} = \omega_n x_n. \tag{38}$$

The time evolution of the system can be expressed in terms of the evolution operator $\mathbf{U}(t)$, defined by the relation

$$\begin{pmatrix} x_0(t) \\ \vdots \\ x_{2N+1}(t) \end{pmatrix} = \mathbf{U}(t) \begin{pmatrix} x_0(0) \\ \vdots \\ x_{2N+1}(0) \end{pmatrix}. \tag{39}$$

Let us introduce the $(2N+2) \times (2N+2)$ matrices $\mathbf{A}$ and $\mathbf{B}$, defined in block form as

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & -\mathbf{\Omega} \\ \mathbf{\Omega} & \mathbf{0} \end{pmatrix} \qquad \text{and} \qquad \mathbf{B}(t) = \begin{pmatrix} \mathbf{0} & \mathbf{\Omega}^{-1/2}\mathbf{W}(t)\mathbf{\Omega}^{-1/2} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{W}(t)$ is the $(N+1) \times (N+1)$ matrix whose elements are defined by equation (20) and $\mathbf{\Omega}$ is the $(N+1) \times (N+1)$ diagonal matrix:

$$\mathbf{\Omega}_{nk} = \omega_n \delta_{nk}.$$

Writing the dynamical equations (38) in the matrix form and taking into account equation (39), it is easy to see that the evolution operator is the solution of the matrix differential equation

$$\dot{\mathbf{U}} = [\mathbf{A} + \varepsilon \mathbf{B}(t)]\mathbf{U},$$

with the initial condition $\mathbf{U}(0) = \mathbf{1}$. Going to the interaction representation, one can write the evolution operator in the form

$$\mathbf{U}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{U}_I(t),$$

where the first factor is the unperturbed evolution operator,

$$\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} \cos \mathbf{\Omega} t & -\sin \mathbf{\Omega} t \\ \sin \mathbf{\Omega} t & \cos \mathbf{\Omega} t \end{pmatrix},$$

while the second factor obeys the equation

$$\dot{\mathbf{U}}_I = \varepsilon \mathbf{B}_I(t)\mathbf{U}_I \tag{40}$$

with

$$\mathbf{B}_I(t) = \mathrm{e}^{-\mathbf{A}t}\mathbf{B}(t)\,\mathrm{e}^{\mathbf{A}t}.$$

For our purposes it is useful to write the evolution operator in block form:

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}^{(a)} & \mathbf{U}^{(c)} \\ \mathbf{U}^{(b)} & \mathbf{U}^{(d)} \end{pmatrix} \qquad \text{and} \qquad \mathbf{U}_I = \begin{pmatrix} \mathbf{U}_I^{(a)} & \mathbf{U}_I^{(c)} \\ \mathbf{U}_I^{(b)} & \mathbf{U}_I^{(d)} \end{pmatrix}.$$

This decomposition allows one to obtain from equation (40) the dynamical equations for the left blocks of the evolution operator in interaction representation

$$\begin{aligned} \dot{\mathbf{U}}_I^{(a)} &= \varepsilon \mathbf{W}_{cs}(t)\mathbf{U}_I^{(a)} + \varepsilon \mathbf{W}_{cc}(t)\mathbf{U}_I^{(b)} \\ \dot{\mathbf{U}}_I^{(b)} &= -\varepsilon \mathbf{W}_{ss}(t)\mathbf{U}_I^{(a)} - \varepsilon \mathbf{W}_{sc}(t)\mathbf{U}_I^{(b)}, \end{aligned} \tag{41}$$

where we have introduced the new symbols

$$\mathbf{W}_{cc}(t) = \cos\boldsymbol{\Omega}t\,\boldsymbol{\Omega}^{-1/2}\mathbf{W}(t)\boldsymbol{\Omega}^{-1/2}\cos\boldsymbol{\Omega}t$$
$$\mathbf{W}_{cs}(t) = \cos\boldsymbol{\Omega}t\,\boldsymbol{\Omega}^{-1/2}\mathbf{W}(t)\boldsymbol{\Omega}^{-1/2}\sin\boldsymbol{\Omega}t$$
$$\mathbf{W}_{sc}(t) = \sin\boldsymbol{\Omega}t\,\boldsymbol{\Omega}^{-1/2}\mathbf{W}(t)\boldsymbol{\Omega}^{-1/2}\cos\boldsymbol{\Omega}t$$
$$\mathbf{W}_{ss}(t) = \sin\boldsymbol{\Omega}t\,\boldsymbol{\Omega}^{-1/2}\mathbf{W}(t)\boldsymbol{\Omega}^{-1/2}\sin\boldsymbol{\Omega}t.$$

The identity $\mathbf{U}_I(0) = \mathbf{1}$ implies that equations (41) are to be solved with initial conditions

$$\mathbf{U}_I^{(a)}(0) = \mathbf{1} \qquad \text{and} \qquad \mathbf{U}_I^{(b)}(0) = \mathbf{0}. \tag{42}$$

We can now select as initial parallelepiped the $(N+1)$-dimensional cube of edge Δ in the space of rescaled momenta. In other words, we consider the set of initial vectors

$$\vec{\xi}_k^{(i)}(0) = \Delta\delta_{ik},$$

where the index i (which identifies the independent vectors) runs from 0 to N, while the index k (which labels the components of each vector) runs from 0 to $2N+1$. With this choice of the initial vectors, and remembering that the evolved vectors can be written as $\vec{\xi}^{(i)}(t) = \mathbf{U}(t)\vec{\xi}^{(i)}(0)$, one obtains that matrix (36) can be expressed in terms of the left blocks of the evolution operator

$$\mathbf{G} = \big[\mathbf{U}^{(a)\,T}\mathbf{U}^{(a)} + \mathbf{U}^{(b)\,T}\mathbf{U}^{(b)}\big]\Delta^2 = \big[\mathbf{U}_I^{(a)\,T}\mathbf{U}_I^{(a)} + \mathbf{U}_I^{(b)\,T}\mathbf{U}_I^{(b)}\big]\Delta^2,$$

(where the symbol $\mathbf{M}^T$ denotes the transpose of the matrix $\mathbf{M}$). Note that the matrix $\mathbf{G}$ has the same form in the interaction representation and in the original representation. Inserting this matrix in formula (35), one obtains the volume of the expanding parallelepiped; after substituting this result in expression (37) one arrives at the conclusion that the sum of the positive Lyapunov exponents can be written as

$$\begin{aligned}\sum_{i=0}^{N}\lambda_i &= \lim_{T\to\infty}\frac{1}{2T}\big\langle \log\det\big[\mathbf{U}_I^{(a)\,T}(T)\mathbf{U}_I^{(a)}(T) + \mathbf{U}_I^{(b)\,T}(T)\mathbf{U}_I^{(b)}(T)\big]\big\rangle \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_0^T \mathrm{d}t\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\log\det\big[\mathbf{U}_I^{(a)\,T}(t)\mathbf{U}_I^{(a)}(t) + \mathbf{U}_I^{(b)\,T}(t)\mathbf{U}_I^{(b)}(t)\big]\right\rangle.\end{aligned}$$

Using the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\det\mathbf{M} = \mathrm{Tr}(\dot{\mathbf{M}}\mathbf{M}^{-1}),$$

one can express the Kolmogorov entropy in the form

$$\begin{aligned}\sum_{i=0}^{N}\lambda_i = \lim_{T\to\infty}\frac{1}{2T}\int_0^T \mathrm{d}t\big\langle \varepsilon\,\mathrm{Tr}\big\{&\big[\mathbf{W}_{cs}(t) + \mathbf{W}_{cs}^T(t)\big]\mathbf{Z}_1(t) - \big[\mathbf{W}_{sc}(t) + \mathbf{W}_{sc}^T(t)\big]\mathbf{Z}_2(t) \\ &+ \big[\mathbf{W}_{cc}(t) - \mathbf{W}_{ss}^T(t)\big]\mathbf{Z}_3(t) + \big[\mathbf{W}_{cc}^T(t) - \mathbf{W}_{ss}(t)\big]\mathbf{Z}_3^T(t)\big\}\big\rangle,\end{aligned} \tag{43}$$

where we have introduced the operators

$$\begin{aligned}\mathbf{Z}_1 &= \mathbf{U}_I^{(a)}\big[\mathbf{U}_I^{(a)\,T}\mathbf{U}_I^{(a)} + \mathbf{U}_I^{(b)\,T}\mathbf{U}_I^{(b)}\big]^{-1}\mathbf{U}_I^{(a)\,T} \\ \mathbf{Z}_2 &= \mathbf{U}_I^{(b)}\big[\mathbf{U}_I^{(a)\,T}\mathbf{U}_I^{(a)} + \mathbf{U}_I^{(b)\,T}\mathbf{U}_I^{(b)}\big]^{-1}\mathbf{U}_I^{(b)\,T} \\ \mathbf{Z}_3 &= \mathbf{U}_I^{(b)}\big[\mathbf{U}_I^{(a)\,T}\mathbf{U}_I^{(a)} + \mathbf{U}_I^{(b)\,T}\mathbf{U}_I^{(b)}\big]^{-1}\mathbf{U}_I^{(a)\,T}.\end{aligned} \tag{44}$$

Taking into account the dynamical equations (41) for the blocks of the evolution operator, it is easy to see that the operators (44) satisfy the differential equations

$$\begin{aligned}
\dot{\mathbf{Z}}_1 &= \varepsilon\big[\mathbf{W}_{cs}\mathbf{Z}_1 + \mathbf{Z}_1\mathbf{W}_{cs}^T + \mathbf{W}_{cc}\mathbf{Z}_3 + \mathbf{Z}_3^T\mathbf{W}_{cc}^T - \mathbf{Z}_1\big(\mathbf{W}_{cs} + \mathbf{W}_{cs}^T\big)\mathbf{Z}_1 \\
&\qquad + \mathbf{Z}_3^T\big(\mathbf{W}_{sc} + \mathbf{W}_{sc}^T\big)\mathbf{Z}_3 - \mathbf{Z}_1\big(\mathbf{W}_{cc} - \mathbf{W}_{ss}^T\big)\mathbf{Z}_3 - \mathbf{Z}_3^T\big(\mathbf{W}_{cc}^T - \mathbf{W}_{ss}\big)\mathbf{Z}_1\big] \\
\dot{\mathbf{Z}}_2 &= \varepsilon\big[-\mathbf{W}_{sc}\mathbf{Z}_2 - \mathbf{Z}_2\mathbf{W}_{sc}^T - \mathbf{W}_{ss}\mathbf{Z}_3^T - \mathbf{Z}_3\mathbf{W}_{ss}^T + \mathbf{Z}_2\big(\mathbf{W}_{sc} + \mathbf{W}_{sc}^T\big)\mathbf{Z}_2 \\
&\qquad - \mathbf{Z}_3\big(\mathbf{W}_{cs} + \mathbf{W}_{cs}^T\big)\mathbf{Z}_3^T - \mathbf{Z}_3\big(\mathbf{W}_{cc} - \mathbf{W}_{ss}^T\big)\mathbf{Z}_2 - \mathbf{Z}_2\big(\mathbf{W}_{cc}^T - \mathbf{W}_{ss}\big)\mathbf{Z}_3^T\big] \\
\dot{\mathbf{Z}}_3 &= \varepsilon\big[-\mathbf{W}_{ss}\mathbf{Z}_1 - \mathbf{W}_{sc}\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{W}_{cs}^T + \mathbf{Z}_2\mathbf{W}_{cc}^T - \mathbf{Z}_3\big(\mathbf{W}_{cs} + \mathbf{W}_{cs}^T\big)\mathbf{Z}_1 \\
&\qquad + \mathbf{Z}_2\big(\mathbf{W}_{sc} + \mathbf{W}_{sc}^T\big)\mathbf{Z}_3 - \mathbf{Z}_3\big(\mathbf{W}_{cc} - \mathbf{W}_{ss}^T\big)\mathbf{Z}_3 - \mathbf{Z}_2\big(\mathbf{W}_{cc}^T - \mathbf{W}_{ss}\big)\mathbf{Z}_1\big].
\end{aligned} \tag{45}$$

These equations determine the time evolution of the operators (44) together with the initial conditions

$$\mathbf{Z}_1(0) = \mathbf{1}, \qquad \mathbf{Z}_2(0) = \mathbf{0}, \qquad \mathbf{Z}_3(0) = \mathbf{0},$$

which can be obtained by substituting in definitions (44) the initial conditions (42) for the left blocks of the evolution operator. The system of equations (45) can be solved perturbatively by considering solutions of the form

$$\mathbf{Z}_i(t) = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{Z}_i^{(n)}(t).$$

Substituting these trial solutions into equation (45), one obtains the simple result

$$\mathbf{Z}_1(t) = \mathbf{1} + o(\varepsilon) \qquad \mathbf{Z}_2(t) = o(\varepsilon) \qquad \mathbf{Z}_3(t) = -\varepsilon \int_0^t \mathbf{W}_{ss}(\tau)\,\mathrm{d}\tau + o(\varepsilon).$$

Putting these expressions in equation (43) one obtains that the sum of the positive Lyapunov exponents is

$$\begin{aligned}
\sum_{i=0}^{N} \lambda_i = \sum_{n=0}^{N}\sum_{k=0}^{N} \frac{\varepsilon^2}{8\omega_n\omega_k} \int_0^{\infty} \{&[\langle W_{kn}(t)W_{kn}(t+\tau)\rangle + \langle W_{nk}(t)W_{kn}(t+\tau)\rangle]\cos(\omega_n + \omega_k)\tau \\
&+ [\langle W_{kn}(t)W_{kn}(t+\tau)\rangle - \langle W_{nk}(t)W_{kn}(t+\tau)\rangle]\cos(\omega_n - \omega_k)\tau\}\,\mathrm{d}\tau + o(\varepsilon^2).
\end{aligned} \tag{46}$$

This formula gives the sum of the positive Lyapunov exponents for the dynamical system (19). Note that the result has been derived without considering any specific form of the coupling matrix W_{nk}. Therefore it can be applied also to cases in which the coupling matrix differs from the form (20). In particular, if one considers a symmetric matrix, $W_{nk} = W_{kn}$, the second term in the rhs of equation (46) vanishes and the result reduces to the form obtained in [15] for a similar problem. Here, however, the W_{nk} matrix is not fully symmetric, because of the Kronecker delta in equation (20) which can be traced back to the fact that the zeroth Fourier mode has no twin component unlike the other modes (which come in equal pairs $\tilde{U}_n = \tilde{U}_{-n}$). However, the second term in equation (46) can be neglected in the limit of a large number of Fourier components, in which case the asymmetry linked to the zeroth channel becomes negligible.

To analyse the physical implications of equation (46), it is useful to express the Kolmogorov entropy in terms of the Fourier components of the random potential U.

Substituting the matrix elements (20) in equation (46), one obtains

$$\sum_{i=0}^{N}\lambda_i = \sum_{n=1}^{N}\frac{1}{8\omega_n\omega_0}\int_0^\infty \langle\tilde{U}_n(t)\tilde{U}_n(t+\tau)\rangle[\cos(\omega_n+\omega_0)\tau+\cos(\omega_n-\omega_0)\tau]\,\mathrm{d}\tau$$
$$+\sum_{n=-N}^{N}\sum_{k=-N}^{N}\frac{1}{8\omega_n\omega_k}\int_0^\infty[\langle\tilde{U}_{n+k}(t)\tilde{U}_{n+k}(t+\tau)\rangle+\langle\tilde{U}_{n-k}(t)\tilde{U}_{n+k}(t+\tau)\rangle]$$
$$\times\cos(\omega_n+\omega_k)\tau\,\mathrm{d}\tau. \tag{47}$$

Note that we have set the book-keeping parameter $\varepsilon = 1$, as we will do from now on, with the tacit understanding that all results are valid within the second-order approximation. Equation (47) represents the Kolmogorov entropy for any kind of random potential. We observe that the second sum on the rhs of equation (47) contains a number of terms of order $O(N^2)$ and therefore for $N \gg 1$ it is dominant with respect to the first sum which has only $O(N)$ terms. Hence if the number of oscillators/modes N is large, one can approximate the Kolmogorov entropy (47) with

$$\sum_{i=0}^{N}\lambda_i \simeq \sum_{n=-N}^{N}\sum_{k=-N}^{N}\frac{1}{8\omega_n\omega_k}\int_0^\infty[\langle\tilde{U}_{n+k}(t)\tilde{U}_{n+k}(t+\tau)\rangle+\langle\tilde{U}_{n-k}(t)\tilde{U}_{n+k}(t+\tau)\rangle]$$
$$\times\cos(\omega_n+\omega_k)\tau\,\mathrm{d}\tau. \tag{48}$$

We remember that the binary correlators which appear in equation (48) are the correlators of the Fourier components (10) of the random potential. They are linked to the binary correlator (6) via the double Fourier transform

$$\langle\tilde{U}_n(t)\tilde{U}_k(t+\tau)\rangle = \frac{\sigma^2}{4L^2}\int_{-L}^{L}\mathrm{d}y\int_{-L}^{L}\mathrm{d}y'\,\chi(\tau,y-y')\cos\frac{\pi ny}{L}\cos\frac{\pi ky'}{L}. \tag{49}$$

If the correlators (49) decay sufficiently fast as a function of the difference $|n-k|$ of the indices of the Fourier components of the potential, the second correlator in equation (48) gives a marginal contribution with respect to the first one. Therefore one can replace expression (48) with

$$\sum_{i=0}^{N}\lambda_i \simeq \sum_{n=-N}^{N}\sum_{k=-N}^{N}\frac{1}{8\omega_n\omega_k}\int_0^\infty\langle\tilde{U}_{n+k}(t)\tilde{U}_{n+k}(t+\tau)\rangle\cos(\omega_n+\omega_k)\tau\,\mathrm{d}\tau. \tag{50}$$

5.2. Application to specific cases

We will now apply the general formulae derived in the previous subsection to a few particular cases and see how one can recover known specific results from the general expressions (47) and (50). Let us consider first the case of longitudinal disorder, i.e., of a random potential of the form $U(x,y) = U(x)$. In this case the only non-vanishing Fourier component (10) is the zeroth one,

$$\tilde{U}_n(t) = \delta_{n0}U(t),$$

and the binary correlators (49) become

$$\langle\tilde{U}_n(t)\tilde{U}_k(t+\tau)\rangle = \sigma^2\chi(\tau)\delta_{n0}\delta_{k0}. \tag{51}$$

These correlators vanish unless $n = k = 0$, and therefore there is no doubt that they decay fast for increasing values of $|n-k|$. Hence the approximate formula (50) can be applied; inserting expression (51) in equation (50) one obtains

$$\sum_{n=0}^{N}\lambda_n \simeq \sum_{n=0}^{N}\frac{\sigma^2}{4\omega_n^2}\int_0^\infty\chi(\tau)\cos 2\omega_n\tau\,\mathrm{d}\tau\left(1-\frac{1}{2}\delta_{n0}\right). \tag{52}$$

The exact result can be obtained by substituting the correlators (51) in equation (47); this gives

$$\sum_{n=0}^{N} \lambda_n = \sum_{n=0}^{N} \frac{\sigma^2}{4\omega_n^2} \int_0^\infty \chi(\tau) \cos 2\omega_n \tau \, \mathrm{d}\tau, \tag{53}$$

and by comparing equation (52) with equation (53) we see that the difference between the two expressions is indeed negligible in the large N limit. We remark that equation (53) is in perfect agreement with what one obtains for the sum of the positive Lyapunov exponents (31) derived in section 4. From the physical point of view, equation (53) is a natural consequence of the fact that, as discussed in section 4, in the case of longitudinal disorder the Fourier modes/oscillators are decoupled so that the quasi-1D model reduces to a sum of strictly 1D systems.

As a special application of formula (53) one can consider the case of longitudinal white noise. In this case $\chi(\tau) = \delta(\tau)$ and the Kolmogorov entropy (53) takes the form

$$\sum_{n=0}^{N} \lambda_n = \sum_{n=0}^{N} \frac{\sigma^2}{8\omega_n^2}$$

in agreement with the well-known expression

$$\lambda_n = \frac{\sigma^2}{8\omega_n^2}$$

for the Lyapunov exponent of a noisy 1D oscillator of frequency ω_n.

Equation (53) can be used also to derive the inverse localization length for a 1D model, which can be seen as the limit case of a quasi-1D model when the transversal dimensions go to zero, $L \to 0$. In this limit the only elliptic Fourier component is the zeroth one and, indeed, setting $N = 0$ in equation (53) one recovers the well-known expression (3) for the inverse localization length in 1D models.

Let us now consider the case of a random potential which is a white noise in both the longitudinal and the transversal directions. In this case the binary correlator (6) has the form

$$\chi(x, y) = \delta(x)\delta(y),$$

and the correlators of the Fourier components (10) are

$$\langle \tilde{U}_n(t) \tilde{U}_k(t+\tau) \rangle = \frac{\sigma^2}{4L} \delta(\tau)(\delta_{n,k} + \delta_{n,-k}). \tag{54}$$

The form of the correlators (54) implies that the second correlator in expression (48) gives only a marginal contribution, so we can substitute equation (54) in formula (50). Neglecting terms with $O(N)$ addends, one obtains

$$\sum_{i=0}^{N} \lambda_i \simeq \frac{\sigma^2}{16L} \left(\sum_{n=0}^{N} \frac{1}{\omega_n} \right)^2. \tag{55}$$

This equation coincides with the result obtained for a discrete model by Hansel and Luciani in [14] if one identifies the width $2L$ of the doubled strip and the frequencies ω_n in (55), respectively, with the number of channels and the square roots of the energies in the model of Hansel and Luciani.

5.3. A sufficient condition for delocalization

The importance of equation (47) lies in the fact that it provides a criterion for the onset of a delocalization transition in quasi-1D models. In fact, the condition

$$\sum_{i=0}^{N} \lambda_i = 0 \tag{56}$$

represents both a necessary and sufficient condition for the suppression of the orbit instability in the system of parametric oscillators (19) and a sufficient condition for the localization length to diverge. That condition (56) is not necessary for the delocalization of the electronic states depends on the fact that, as discussed in section 2.6, the localization length is equal to the *smallest* Lyapunov exponent. Therefore, delocalization sets in as soon as the minimum Lyapunov exponent vanishes, even if larger Lyapunov exponents are non-zero. In contrast, in the case of the dynamical system (19), unless *all* Lyapunov exponents vanish, initially nearby orbits exponentially diverge.

The importance of expression (47) rests on the fact that it allows one to prove that, for specific kinds of long-range correlated disorder, condition (56) is fulfilled over a certain range of the electronic energy and, therefore, a continuum of extended states arises even in quasi-1D models. The effect is analogous to that observed in strictly 1D models [2, 12]. To see how a delocalization transition can occur, one can observe that the Kolmogorov entropy (47) vanishes if the Fourier transforms in the longitudinal direction of the binary correlators (49) are zero,

$$\int_0^\infty \left\langle \tilde{U}_{n_1}(t)\tilde{U}_{n_2}(t+\tau)\right\rangle \cos\omega\tau \, \mathrm{d}\tau = 0, \tag{57}$$

for all values of the indices n_1 and n_2 and for every frequency ω in the interval $[0 : 2\sqrt{E}]$. The frequency interval is determined by taking into account that the frequencies (15) vary in the interval $[0 : \sqrt{E}]$ and that formula (47) contains cosines with frequency up to twice the maximum value of the frequencies ω_n. Condition (57), however, is ensured by requiring that the Fourier transform of the correlator (6)

$$\tilde{\chi}(\omega_x, \omega_y) = \int_0^\infty \mathrm{d}x \int_0^L \mathrm{d}y \, \chi(x, y) \cos\omega_x x \cos\omega_y y \tag{58}$$

should vanish for $0 \leqslant \omega_x \leqslant 2\sqrt{E}$, i.e.,

$$\tilde{\chi}(\omega_x, \omega_y) = 0 \qquad \text{for} \quad 0 \leqslant \omega_x \leqslant 2\sqrt{E}. \tag{59}$$

A potential whose binary correlator satisfies this condition can be constructed with a slight generalization of the method discussed in section 4 for the longitudinal disorder case. Starting from a binary correlator (58) with arbitrary frequency dependence, one can obtain the function

$$\beta(x, y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega_x}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega_y}{2\pi} \sqrt{\tilde{\chi}(\omega_x, \omega_y)} \exp(-\mathrm{i}\omega_x x - \mathrm{i}\omega_y y).$$

The required potential can then be constructed via the convolution product

$$U(x, y) = \sigma^2 \int_{-\infty}^{\infty} \mathrm{d}s_x \int_{-\infty}^{\infty} \mathrm{d}s_y \, \beta(s_x, s_y)\eta(s_x + x, s_y + y),$$

where η is a stochastic process with

$$\langle \eta(x, y)\rangle = 0 \qquad \text{and} \qquad \langle \eta(x, y)\eta(x', y')\rangle = \delta(x - x')\delta(y - y').$$

Following this procedure, one can obtain a potential which fulfils the delocalization condition (59). For example, one can consider a potential with long-range correlations in the longitudinal direction of the form

$$\chi(x, y) = \frac{1}{x}(\sin\nu_1 x - \sin\nu_2 x)\gamma(y), \tag{60}$$

where $\gamma(y)$ is the correlation function in the transversal direction. The binary correlator (60) is the simplest possible generalization of the correlator (34). Its power spectrum has the form

$$\tilde{\chi}(\omega_x, \omega_y) = \begin{cases} \tilde{\gamma}(\omega_y) & \text{if } \nu_1 < \omega_x < \nu_2 \\ 0 & \text{otherwise.} \end{cases}$$

By selecting a potential such that $\nu_1 = 2\sqrt{E}$, one is ensured that the delocalization condition (56) is fulfilled. We are thus led to the conclusion that a delocalization transition can occur if the disorder exhibits long-range correlations that make the power spectrum vanish in an appropriate frequency interval.

One should hasten to add that this conclusion is rigorously true only for weak disorder and within the second-order approximation. In 1D models it has been shown that the delocalization transition produced by long-range correlations of the disorder is a second-order effect and the 'extended states' are, in fact, electronic states which extend over a spatial range of order $O(1/\sigma^4)$ rather than $O(1/\sigma^2)$ [13]. The same result can be expected for quasi-1D models; this does not diminish the practical importance of the delocalization transition analysed here because, for weak disorder, i.e., when $\sigma^2 \to 0$, the increase of the spatial range of the electronic wavefunction can be huge, being of order $O(1/\sigma^2)$. For finite samples, therefore, the delocalization can be real and manifest itself in a strong change of the transport properties of the disordered sample.

6. Conclusions

In this work we have shown how the spatial structure of electronic states in a quantum quasi-1D model with weak disorder can be analysed in terms of the time evolution of a classical system of parametric oscillators with weak stochastic couplings. By Fourier transforming the stationary Schrödinger equation for the quasi-1D model in the transversal directions, one obtains a set of equations for the Fourier components of the electronic wavefunction that can be mapped unto the dynamical equations of a Hamiltonian system of coupled oscillators. The spatial behaviour of the Fourier components of the wavefunction is thus matched to the time evolution of the oscillators, while the disorder in the quasi-1D model manifests itself as noise in its dynamical analogue. The specific effect of the noise is to perturb the frequencies of the oscillators and to couple the oscillators among themselves. Both models are characterized by a set of characteristic Lyapunov exponents; however, whereas in the solid state model the key Lyapunov exponent is the smallest one, which is equal to the inverse of the localization length, in the dynamical system the most important exponent is the largest, which defines the mean rate of exponential divergence of the orbits.

The dynamics of the Hamiltonian system can be analysed in full detail when the random potential in the quasi-1D model depends only on the longitudinal coordinate. In this case the oscillators are effectively decoupled and it is possible to obtain the whole Lyapunov spectrum of the system. One can thus make a complete study of the delocalization effects produced by specific long-range correlation of the disorder.

The general case of a random potential which depends both on longitudinal and transversal coordinates is more difficult to handle; nevertheless, it is still possible to evaluate perturbatively the sum of the positive Lyapunov exponents (or Kolmogorov entropy). Using this result, one can show that specific kinds of long-range correlated disorder make all Lyapunov exponents vanish within the second-order approximation and therefore produce a delocalization transition in quasi-1D models like they do in strictly 1D models.

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